Physical Properties of Small DC Motors Using an ironless Rotor

A technical publication about escap® products
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In modern technical developments of all kinds we find an increasing number of servo systems where quite often motors are used either as a sensor or as an actuator. For these applications d.c. motors with an ironless armature and permanent magnet are particularly well suited, due to their unique linear characteristics. Low armature inertia and very low starting voltages are additional decisive advantages with this kind of construction. Since the introduction of the escap\textsuperscript{®} motor in 1960, we have time and again been asked questions concerning the physical properties of this motor. Therefore we thought it useful to give a brief, yet thorough, explanation of these properties. The simple mathematical relations have been illustrated by some practical calculations.
## Signs and units of the physical magnitudes used

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<tr>
<th>Symbol</th>
<th>Description</th>
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<tr>
<td>I</td>
<td>armature current</td>
<td>A (Ampere)</td>
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<tr>
<td>I_0</td>
<td>no-load current</td>
<td>A</td>
</tr>
<tr>
<td>I_s</td>
<td>starting current</td>
<td>A</td>
</tr>
<tr>
<td>J_0</td>
<td>moment of inertia of rotor</td>
<td>Kg m²</td>
</tr>
<tr>
<td>J_L</td>
<td>load inertia referred to motor shaft</td>
<td>Kg m²</td>
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<tr>
<td>k</td>
<td>motor constant</td>
<td>Nm/s/A = Vs</td>
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<tr>
<td>k_r</td>
<td>torque constant</td>
<td>Nm/A</td>
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<td>torque developed in the armature</td>
<td>Nm (Newtonmeter)</td>
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<td>M_st</td>
<td>stall torque</td>
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<td>M_fr</td>
<td>friction torque of running motor</td>
<td>Nm</td>
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<tr>
<td>M_L</td>
<td>load torque</td>
<td>Nm</td>
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<td>n</td>
<td>speed (in revolutions per minute)</td>
<td>rpm</td>
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<tr>
<td>n_0</td>
<td>no-load speed</td>
<td>rpm</td>
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<td>P_e</td>
<td>electrical power</td>
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<td>P_m</td>
<td>mechanical power output</td>
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<td>Q</td>
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<td>armature resistance at 22°C</td>
<td>Ω</td>
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<td>U_p</td>
<td>power supply voltage</td>
<td>V [Volt]</td>
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<tr>
<td>U_i</td>
<td>voltage induced in the rotor (back-emf)</td>
<td>V</td>
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<tr>
<td>a</td>
<td>angular acceleration</td>
<td>rad/s²</td>
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<tr>
<td>γ</td>
<td>thermal coefficient of the resistance of copper</td>
<td>0.0041°C</td>
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<tr>
<td>ΔT</td>
<td>heating (temperature rise)</td>
<td>°C</td>
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<tr>
<td>η</td>
<td>efficiency</td>
<td>s [seconds]</td>
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<tr>
<td>τ</td>
<td>time constant</td>
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<td>τ_m</td>
<td>mechanical time constant of the unloaded motor</td>
<td>rad [radians]</td>
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<tr>
<td>δ</td>
<td>angular displacement of rotor (δt = 0) = 0</td>
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<tr>
<td>ω</td>
<td>angular speed</td>
<td>rad/s</td>
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<tr>
<td>ω_0</td>
<td>no-load angular speed</td>
<td></td>
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### Conversion factors
- 1 oz.in. = 7.08 mNm
- 1 oz.in. s² = 7.08 × 10⁻³ kg m²
- 1 kcal = 4186 Ws
Construction of the escap® motor

Direct-current micromotors with ironless rotors are formed of 3 elements: the magnetic system, the ironless rotor and the endcap with the brushes. The magnetic system consists of a cylindrical core, fitted by means of its front plate to the tubular housing; the housing closes the magnetic circuit. The two bearings are either of the self-lubricated sintered bronze type or ball bearings; one is located at the centre of the magnetic core, while the other is mounted at the centre of the front plate, which generally has tapped holes for the fixing screws. The magnetic core is magnetized diametrically after it has been mounted in the magnetic system. Owing to the low permeability of the permanent magnets, the magnetic resistance is great enough to allow neglect of the weakening of the stator field due to the rotor current. The ironless rotor, in the form of a bell, rotates round the magnetic core. In escap® motors, the rotor is made in the form of a skew wound coil and is fixed to a disc carrying the commutator. Depending on the motor dimensions, the commutator has between 5 and 13 segments. The small commutator diameter results in low sliding-speed and the low friction gives high efficiency. The perfectly symmetrical construction of the skew wound coil makes for smooth and silent running. The crossing of the wires makes the self supporting rotor winding (armature) highly stable mechanically. The endcap includes the brush system. Owing to the use of precious metals for the brushes and the commutator segments, the low contact resistance of the brushes is maintained throughout the life of the micromotor and ensures starting at very low voltages. For motors operating continually at high currents the Rotafe® copper/graphite commutation system is also available.
Fundamental equations

The electromechanical properties of motors with ironless rotors can be described by means of the following equations:

1. The power supply voltage $U_i$ is equal to the sum of the voltage drop produced by the current $I$ in the ohmic resistance $R_m$ of the rotor winding, and the voltage $U_i$ induced in the rotor:

$$U_0 = I \times R_m + U_i \quad \text{(1)}$$

2. The voltage $U_i$ induced in the rotor is proportional to the angular velocity $\omega$ of the rotor:

$$U_i = k_i \times \omega \quad \text{(2)}$$

It should be noted that the following relationship exists between the angular velocity $\omega$ expressed in radians per second and the speed of rotation $n$ expressed in revolutions per minute:

$$\omega = \frac{2 \pi \times n}{60}$$

3. The rotor torque $M$ is proportional to the rotor current $I$:

$$M = k_r \times I \quad \text{(3)}$$

By substituting the fundamental equations (2) and (3) into (1), we obtain the characteristic of torque/angular velocity for a d.c. motor with an ironless rotor:

$$U_0 = M \times \frac{R_m}{k_t} + k_s \times \omega \quad \text{(4)}$$

By calculating the constant $k_t$ and $k_r$ from the dimensions of the motor, the number of turns per winding, the number of windings, the diameter of the rotor and the magnetic field in the air gap, we find for the direct-current micromotor with an ironless rotor:

$$M = \frac{U_0}{\omega} = k \quad \text{(5)}$$

which means that $k = k_s = k_r$.

The identity $k_s = k_r$ is also apparent from the following energetic considerations:

The electric power $P_e = U_i \times I$ which is supplied to the motor must be equal to the sum of the mechanical power $P_m = M \times \omega$ produced by the rotor and the dissipated power (according to Joule's law) $P_d = I^2 \times R_m$:

$$P_e = P_m + P_d$$

or:

$$U_i \times I = M \times \omega + I^2 \times R_m$$

Moreover, by multiplying equation (1) by $I$, we also obtain a formula for the electric power $P_e$:

$$P_e = U_i \times I = I^2 \times R_m + U_i \times I$$

The equivalence of the two equations gives:

$$M \times \omega = U_i \times I$$

or:

$$\frac{U_i}{\omega} = \frac{M}{I}$$

and $k_s = k_t = k_r$

quod erat demonstrandum.

Using the above relationships, we may write the fundamental equations (1) and (2) as follows:

$$U_0 = I \times R_m + k \times \omega \quad \text{(6)}$$

and:

$$U_0 = M \times \frac{R_m}{k} + k \times \omega \quad \text{(7)}$$

Graphic expression "speed-torque" characteristic:

To overcome the friction torque $M_f$ due to the friction of the brushes and bearings, the motor consumes a no-load current $I_0$.

This gives:

$$M_f = k \times I_0$$

and:

$$U_0 = I_0 \times R_m + k \times \omega_0 \quad \text{where} \quad \omega_0 = \frac{2 \pi \times n_0}{60}$$

hence:

$$k = \frac{U_0 - I_0 \times R_m}{\omega_0} \quad \text{(8)}$$

It is therefore perfectly possible to calculate the motor constant $k$ with the no-load speed $n_0$, the no-load current $I_0$ and the rotor resistance $R_m$.

The starting-current $I_s$ is calculated as follows:

$$I_s = \frac{U_0}{R_m}$$

It must be remembered that $R_m$ depends to a great extent on the temperature; in other words, the resistance of the rotor increases with the heating of the motor due to the dissipated power (Joule's law):

$$R_m = R_{m0} (1 + \gamma \times \Delta T)$$

where $\gamma$ is the temperature coefficient of copper $(\gamma = 0.004/\degree C)$.

As the copper mass of the coils is comparatively small, it heats very quickly through the effect of the rotor current, particularly in the event of slow or repeated starting (see chapter "Heating", page 12). The torque $M_s$ produced by the starting-current $I_s$ is obtained as follows:

$$M_s = I_s \times k - M_f = (I_0 - I_s) \frac{M_f}{k} \quad \text{(9)}$$

By applying equation (1), we can calculate the angular velocity $\omega$ produced under a voltage $U_0$ with a load torque $M_l$.

We first determine the current required for obtaining the torque $M = M_l + M_f$:

$$I = \frac{M_l + M_f}{k}$$

Since $\frac{M_f}{k} = I_0$

we may also write

$$I = \frac{M_l}{k} + I_0 \quad \text{(10)}$$

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For the angular velocity \( \omega \) we obtain the relationship

\[
\omega = \frac{U_c - I \times R_M}{k} \quad \text{(11)}
\]

\[
= \frac{U_0}{k} - \frac{R_M}{k^2} (M_l + M_t)
\]

in which the temperature dependence of the rotor resistance \( R_M \) must again be considered; in other words, the value of \( R_M \) at the working temperature of the rotor must be calculated.

On the other hand, with equation (3), we can calculate the current \( I \) and the load torque \( M_l \) for a given angular velocity \( \omega \) and a given voltage \( U_c \):

\[
I = \frac{U_c - k \times \omega}{R_M} = I_d - \frac{k}{R_M} \omega \quad \text{(12)}
\]

and with equation (10)

\[
M_l = (I - I_d) k
\]

we get the value of \( M_l \):

\[
M_l = (I - I_d) k - \frac{k^2}{R_M} \omega
\]

The problem which most often arises is that of determining the power supply voltage \( U_c \) required for obtaining a speed of rotation \( n \) for a given load torque \( M_l \). Angular velocity \( \omega = \frac{n \times 2\pi}{60} \). By introducing equation (10) into (6) we obtain:

\[
U_c = \left( \frac{M_l}{k} + I_d \right) R_M + k \times \omega \quad \text{(13)}
\]

Practical examples of calculations

Please note that the International System of Units (SI) is used throughout.

1. Let us suppose that, for an escap\(^8\) motor 23D21-216E, we wish to calculate the motor constant \( k \), the starting current \( I_0 \) and the starting torque \( M_t \) at a rotor temperature of 40\(^\circ\)C. With a power supply voltage of 12V, the no-load speed \( n_0 \) is 4900 rpm \((\omega_0 = 513 \text{ rad/s})\), the no-load current \( I_0 = 12 \text{ mA} \) and the resistance \( R_W = 9.5 \Omega \) at 22\(^\circ\)C.

By introducing the values \( \omega_0, I_0, R_W \) and \( U_c \) into the equation (8), we obtain the motor constant \( k \) for the motor 23D21-216E:

\[
k = \frac{12 - 0.012 \times 9.5}{513} = 0.0232 \text{ V} / \text{A}
\]

Before calculating the starting-current, we must calculate the rotor resistance at 40\(^\circ\)C. With \( T = 18 \text{ C} \) and \( R_W = 9.5 \Omega \), we obtain:

\[
R_W = (1 + 0.004 \times 18) - 9.5 \times 1.07 - 10.2 \Omega
\]

The starting current \( I_d \) at a rotor temperature of 40\(^\circ\)C becomes

\[
I_d = \frac{U_0}{R_W} = 12 \text{ V} \times 10.2 \Omega = 1.18 \text{ A}
\]

and the starting-torque \( M_t \), according to equation (9), is

\[
M_t = k (I_a - I_d) = 0.0232 (1.18 - 0.012) = 0.027 \text{ Nm}
\]

2. Let us ask the following question: what is the speed of rotation \( n \) attained by the motor with a load torque of 0.008 Nm and a power supply voltage of 9V at a rotor temperature of 40\(^\circ\)C?

Using equation (10), we first calculate the current which is supplied to the motor under these conditions:

\[
I = \frac{M_l}{k} + I_0 = \frac{0.008}{0.0232} + 0.012 = 0.357 \text{ A}
\]

Equation (11) gives the angular velocity \( \omega \):

\[
\omega = \frac{U_c - I \times R_M}{k} = \frac{9 - 0.357 \times 10.2}{0.0232} = 231 \text{ rad/s}
\]

and the speed of rotation \( n \):

\[
n = \frac{\omega}{2\pi} \times 60 = 2200 \text{ rpm}
\]

Thus the motor reaches a speed of 2200 rpm and draws a current of 0.357 mA.

3. Let us now calculate the torque \( M_t \) at a given speed of rotation \( n \) of 3000 rpm \((\omega = 314 \text{ rad/s})\) and a power supply voltage \( U_c \) of 15V, equation (12) gives the value of the current:

\[
I = \frac{U_c - k \times \omega}{R_M} = I_d - \frac{k}{R_M} \times \omega
\]

\[
= 118 - \frac{0.0232 \times 314}{10.2} = 0.486 \text{ A}
\]

and the torque load \( M_t \):

\[
M_t = k (I - I_d) = 0.0232 (0.486 - 0.012) = 0.0105 \text{ Nm}
\]

\[
(M_t = 10.5 \text{ mNm})
\]

4. Lastly, let us determine the power supply voltage \( U_c \) required for obtaining a speed of rotation \( n \) of 4000 rpm \((\omega = 419 \text{ rad/s})\) with a load torque \( M_t \) of 0.008 Nm, the rotor temperature again being 40\(^\circ\)C \((R_W = 10.2 \Omega)\).

As we have already calculated, the current \( I \) necessary for a torque of 0.008 Nm is 0.357 A.

\[
U_c = I \times R_W + k \times \omega
\]

\[
= 0.357 \times 10.2 + 0.0232 \times 419 = 13.4 \text{ volt}
\]
The mechanical power $P_m$ developed by the motor is the product of the torque $M_t$ (load torque) and the angular velocity $\omega$: $P_m = M_t \times \omega$ \hspace{1cm} (14)

First, let us calculate $\omega$ in terms of the load torque $M_t$. Using equation (8) $U_e = R_u \times I_d$, we obtain $R_u \times I_d = I \times R_u + k \times \omega$

or:

$$\omega = \frac{R_u}{k} (I_d - I)$$ \hspace{1cm} (15)

As we can write:

$$I_d = \frac{M_t + M_l}{k}$$

and

$$I = \frac{M_t + M_l}{k}$$

we obtain the equation

$$I_d - I = \frac{1}{k} (M_d - M_l)$$

By introducing this into equation (15), we obtain $\omega$:

$$\omega = \frac{R_u}{k^2} (M_d - M_l)$$

Replacing $\omega$ in equation (14) by this term gives the power $P_m$ as a function of $M_t$:

$$P_m = M_t (M_d - M_l) \frac{R_u}{k^2}$$

$$= (M_t \times M_d - M_t^2) \frac{R_u}{k^2}$$

We obtain the maximum of $P_m$ by setting

$$\frac{dP_m}{dM_t} = 0$$

or:

$$\frac{M_d - 2 M_i}{k^2} \frac{R_u}{k} = 0$$

and therefore:

$$M_i = \frac{M_d}{2}$$

or again:

$$I = \frac{I_d + I_s}{2}$$

In other words, the mechanical power $P_m$ reaches its maximum when the load torque is equal to half the starting torque. The maximum value is:

$$P_m_{max} = \frac{R_u}{k^2} \times \frac{M_d^2}{4}$$

Since $M_0 = (I_d - I_s) k$, this gives:

$$P_m_{max} = \frac{R_u}{4} (I_d - I_s)^2$$ \hspace{1cm} (16)

The electric power $(P_e)_d$ supplied to the motor at the time of starting is:

$$(P_e)_d = U_e \times I_d = R_u \times I_d^2$$

Thus:

$$(P_m)_{max} < \frac{1}{4} (P_e)_d = \frac{U_e^2}{4 R_u}$$ \hspace{1cm} (17)

From the above relationship, it may be deduced that the maximum mechanical power $P_m$ developed by the motor is, at the most, 25% of the electric power supplied to the motor at the time of starting.

This implies that the motor is driven with constant voltage and can draw its full stall current for maximum acceleration. Equation (17) does not apply with a chopper driver.
The efficiency $\eta$ of a motor is defined by the quotient of the mechanical power $P_m$ produced and the electric power $P_e$ supplied:

$$\eta = \frac{P_m}{P_e}$$

For the relationships $M_e$ and $M_t$ obtained in the preceding chapter for the mechanical power, let us substitute:

$$M_e = k (I_e - I_0) \quad \text{and} \quad M_t = k (I - I_0)$$

i.e.

$$M_e - M_t = k (I_e - I)$$

and we obtain for $P_m$:

$$P_m = M_e (M_e - M_t) \frac{R_m}{k^2} = \frac{(I_e - I_0)(I_e - I) R_m}{k^2}$$

Moreover, for the electric power $P_e$ supplied to the motor, we have:

$$P_e = U_e \times I_e = I_e \times I_e \times R_m$$

since:

$$U_e = I_e \times R_m$$

For the efficiency $\eta$, we thus obtain

$$\eta = \frac{P_m}{P_e} = \frac{(I_e - I_0)(I_e - I)}{I_e \times I_e}$$

or:

$$\eta = 1 + \frac{I_0}{I_e} - \frac{I_e}{I_0} - \frac{1}{I_e}$$

(18)

(19)

To determine the maximum value of $\eta (I)$, we take $d \eta/dI = 0$ and obtain:

$$\frac{I_0}{I_e} - \frac{1}{I_e} = 0$$

or $I_e^2 = I_0 \times I_0$ and $I = \sqrt{I_0 \times I_0}$

By introducing the value obtained in the relationship for $\eta (I)$, we obtain:

$$\eta_{max} = \left( 1 - \frac{I_0}{\sqrt{I_0 \times I_0}} \right) \left( 1 - \frac{I_0 \times I_0}{I_0} \right)$$

$$\eta_{max} = \left( 1 - \frac{I_0}{I_0} \right)^2$$

The maximum value $\eta_{max}$ can also be expressed in terms of useful torque. By using the relationships:

$$I_0 = \frac{M_t}{k} \quad \text{and} \quad I_0 = \frac{M_e + M_t}{k}$$

we obtain:

$$\sqrt{\frac{I_0}{I_0}} = \sqrt{\frac{M_t}{M_e + M_t}}$$

and

$$\eta_{max} = \left( 1 - \sqrt{\frac{M_t}{M_t + M_e}} \right)^2$$

(20)

This maximum value is attained with a torque $M = \sqrt{(M_e + M_t) M_t}$ or a load torque

$$M_l = \sqrt{(M_e + M_t) M_t - M_t}$$

$M_t$ is the friction torque of the running motor.

The above relationships demonstrate that the maximum efficiency is a function of the quotient of the starting-current and the no-load current or of the quotient of the starting-torque and the frictional torque. For example, if the no-load current $I_0$ is 1% of the starting current $I_0$, i.e. if:

$$I_0 = \frac{I_0}{100}$$

we obtain for the maximum efficiency:

$$\eta_{max} = \left( 1 - \sqrt{\frac{1}{100}} \right)^2 = (0.9)^2 = 0.81 \quad \text{or} \quad 81\%$$

This value is attained for a current of:

$$I = \frac{I_0 + I_e}{2}$$

by using the equation (16):

$$\eta_{P_m\max} = \frac{R_m}{4} \left( I_e - I_0 \right)^2$$

$$\frac{U_e}{2} \left( I_e + I_0 \right)$$

$$= \frac{1}{2} \left( I_e - I_0 \right)^2$$

$$= \frac{1}{2} \left( I_e \frac{I_0}{I_e} \right)^2 \frac{1}{2} \left( I_e \frac{I_0}{I_e} \right)^2 < \frac{1}{2}$$

Thus, for a motor driven with constant voltage, the efficiency at ($P_m$)$_{max}$ is 50%. However, with a constant current drive (chopper), it may well reach values as high as 80%.
Starting under load

Let us now examine the starting of the motor under load. The moment of inertia of the load referred to the motor shaft is \( J_L \), and the load torque is \( M_L \). The difference between the torque \( M \) produced by the motor and the effective load torque \( M_L \) (plus the friction torque \( M_f \)) accelerates the motor and the load. We thus obtain the following equation of motion:

\[
(M + J_L) \frac{d\omega}{dt} = M - (M_L + M_f)
\]

The torque \( M \) produced by the motor is calculated by means of the basic equation (7):

\[
M = \frac{k}{R_m} (U_e - k \times \omega)
\]

As the mechanical time constant of the motor is

\[
\tau_m = \frac{R_m}{k^2} J_m
\]

we can also formulate:

\[
\tau = \tau_m \left(1 + \frac{J_L}{J_m}\right)
\]

Under the starting condition \( \omega (t = 0) = 0 \), the solution of this differential equation is (see appendix for mathematical development):

\[
\omega = \omega_0 \left(1 - e^{-t/\tau_m}\right)
\]

When the motor is started under load, the angular velocity \( \omega \) increases with time following an exponential function. After the time \( t = \tau \), the motor attains 63% of the angular velocity \( \omega_0 \), which is the velocity attained after an infinite time.

To obtain the starting of the unloaded motor, we take:

\[
M = 0 \quad \text{and} \quad J_L = 0
\]

and obtain:

\[
\omega = \omega_0 \quad \text{or} \quad \omega = \omega_0 \left(1 - e^{-t/\tau_m}\right)
\]

which means that, after infinite time, the motor attains the no-load angular velocity \( \omega_0 \) (no-load speed of rotation \( \omega \)) corresponding to the power supply voltage \( U_e \).

The angular acceleration \( \alpha \) of the unloaded motor with the starting torque \( M_s \) is:

\[
M_s = (l_s - l_l) k = J_m \frac{d\omega}{dt} = J_m \alpha
\]

\[
\alpha = \frac{M_s}{J_m} = \frac{k}{J_m} (l_s - l_l)
\]

The angular acceleration \( \alpha \) is thus proportional to the stall torque \( M_s \) and inverse proportional to the rotor inertia \( J_m \).

If we use equation (11):

\[
\omega_1 = \frac{U_e}{k} - \frac{R_m}{k^2} (M_L + M_f)
\]

i.e. for the angular velocity \( \omega_1 \) of the motor with a supply voltage \( U_e \) and a torque \( M_L \), we can transcribe the above differential equation as follows:

\[
\tau \frac{d\omega}{dt} + \omega = \omega_1 \quad \text{where}
\]

\[
\tau = \frac{R_m}{k^2} (J_m + J_L)
\]

\[
\text{Increase of angular velocity (speed of rotation) with time.}
\]
Examples of calculations

1. Let us calculate the starting time of an escap® motor 23D21–216E with a torque load of Mₐ = 0.008 Nm, a moment of load inertia of Jₙ = 6 × 10⁻⁷ kg m² and a power supply voltage U₀ = 9 V; the motor is required to reach a speed of rotation of n = 1500 rpm (ω₀ = 157 rad/s). The moment of inertia of the rotor is Jₘ = 5.9 × 10⁻⁷ kg m², the motor constant k = 0.0232 Vs, and the resistance Rₘ = 9.5 Ω.

The mechanical time constant τₘ of the motor is

\[ τₘ = \frac{Rₘ}{k^2} \cdot \frac{Jₙ}{Jₘ} = \frac{9.5}{(0.0232)^2} \cdot \frac{5.9 \times 10^{-7}}{6 \times 6.9} = 0.0104 \text{ s} \]

The time constant τ of the loaded motor is calculated as follows:

\[ τ = τₘ \left(1 + \frac{Jₙ}{Jₘ}\right) = 0.0104 \left(1 + \frac{6}{6.9}\right) = 0.021 \text{ s} \]

In our example, we have:

\[ ω₁ = 231 \text{ rad/s and } ω₀ = 157 \text{ rad/s}; \]

therefore:

\[ t = 0.021 \ln \frac{231}{231 - 157} = 0.021 \ln 3.12 = 0.021 \times 1.138 = 0.024 \text{ s} \]

Under the above conditions, the motor 23D21–216E reaches a speed of rotation of 1500 rpm (ω₀ = 157 rad/s) after 24 ms.

2. In a second example, the question is put inversely: what speed of rotation is attained by the unloaded motor 23D21–216E after 30 ms (0.03 s) at a power supply voltage of U₀ = 12 V? As already mentioned, the time constant τₘ of the motor 23D21–216E is equal to 0.0104 s. Using equation (22), we obtain:

\[ ω = ω₀ \left(1 - e^{-t/τₘ}\right) = 513 \left(1 - e^{0.03/0.0104}\right) = 513 \left(1 - 0.60\right) \]

\[ ω = 484 \text{ rad/s, or} \]

\[ n = \frac{60}{2π} \times 484 = 4625 \text{ rpm} \]

The no-load angular velocity ω₀ = 513 rad/s has been calculated in the example on page 5. Thus, after 30 ms (0.03 s), the motor reaches a speed of 4625 rpm.

From:

\[ ω - ω₁ (1 - e^{-t/τₘ}) \]

we derive:

\[ e^{-t/τₘ} = 1 - \frac{ω}{ω₁} = \frac{ω₁ - ω}{ω₁} \]

and:

\[ t = τ \ln \frac{ω₁}{ω₀ - ω} = τ \ln ω₁ - \ln (ω₁ - ω) \]

Therefore:

\[ t = \frac{τ}{ω₁ - ω₀} \]

\[ = \frac{τ \ln ω₁ - \ln (ω₁ - ω₀)}{ω₁ - ω₀} \]

3. The moment of inertia of the rotor of a 23D21–216E is Jₘ = 5.9 × 10⁻⁷ kg m². What will be the initial acceleration a at U₀ = 12 V and at a rotor temperature of 40°C? On page 5 we found the starting current to be Iₘ (12 V) = 118 A and the stall torque Mₘ (12 V) = 0.027 Nm. Using equation (23) we find the initial acceleration at stall to be:

\[ a = \frac{Mₘ}{Jₘ} \]

\[ = \frac{0.027}{5.9 \times 10^{-7}} = 45760 \text{ rad/s}² \]
Starting angle

From the integration of the function \( \omega(t) = \omega_0 (1 - e^{-t/\tau}) \) we obtain the angular displacement \( \delta(t) \), assuming that at the time \( t = 0 \) the angle \( \delta(0) = 0 \).

\[
\delta(t) = \int_0^t \omega_0 (1 - e^{-\tau/\tau}) \, dt = \omega_0 (t - \tau) + \omega_0 \tau e^{-t/\tau}
\]  
(24)

For \( t/\tau > 3 \), we have \( \delta(t) = \omega_0 (t - \tau) \), as \( e^{-t/\tau} \) is only 0.05.
For \( t/\tau < 3 \), we can make a table of \( \delta(t)/(\omega_0 \times \tau) \) in terms of \( t/\tau \). We have:

\[
\frac{\delta(t)}{\omega_0 \times \tau} = \frac{t}{\tau} - 1 + e^{-t/\tau}
\]

The following graph shows the increase of \( \delta(t)/(\omega_0 \times \tau) \) with time:

Examples of calculations

Let us calculate the starting angle \( \delta \) for a motor under the following conditions:
- starting time \( t = 0.05 \) s (50 ms)
- time constant of motor \( \tau_M = 0.02 \) s (20 ms)
- ratio of moment of inertia \( J_L / J_M = 1 \)

The speed of rotation attainable with an load torque \( M_L \) would be \( \omega_0 = 3000 \) rpm \( (\omega_0 = 314 \) rad/s). We first calculate \( \tau \):

\[
\tau = \tau_M \left( 1 + \frac{J_L}{J_M} \right) = 2 \tau_M = 0.04 \text{ s (40 ms)}
\]

Moreover, \( t/\tau = 0.05/0.04 = 1.25 \). For this value, the graph gives \( \delta(t)/(\omega_0 \times \tau) = 0.54 \). This value can also easily be obtained by interpolation between the values for 1.2 and 1.4

Since: \( \omega_0 \times \tau = 314 \times 0.04 = 12.6 \text{ rad} \),
we obtain

for \( \delta \):

\[
\delta = \frac{(\omega_0 \times \tau) \times 0.54}{\omega_0} = 12.6 \times 0.54 = 6.78 \text{ rad}.
\]

If we want to convert this angular displacement \( \delta \) into degrees, we must multiply the above value by \( 180/\pi \) and we obtain:

\[
\delta = 388.6 \text{ degrees} = 1.08 \text{ revolution of the motor shaft}.
\]

If we wish to find the power supply voltage \( U_L \) required for the motor to attain an angular velocity \( \omega_1 \) after a given angular displacement \( \delta_0 \), we must represent \( \omega \) in terms of \( \delta \).

Using the relationships:
\[
\omega_0 = \omega_0 (1 - e^{-t/\tau}) \quad \text{and} \quad \delta_0 = \omega_0 \times \tau - \omega_0 \times \tau
\]
we express \( t \) in terms of \( \delta_0, \omega \) and \( \omega_0 \):

\[
t = \frac{\delta_0 + \omega \times \tau}{\omega_0} = \frac{\omega \times \tau}{\omega_0} \left( \frac{\delta_0}{\omega \times \tau} + 1 \right)
\]
and then introduce it into the equation above:

\[
\frac{\delta_0}{\omega_1} = 1 - e^{-\tau}
\]

The above equation shows that \(\delta_0/\omega_1\) cannot be expressed explicitly in terms of \(\delta_0\). It is however possible to represent \(\delta_0/(\omega \times \tau) + 1\) in terms of \(\omega/\omega_1\).

We have:

\[
A = \frac{\delta_0}{\omega_1} + 1
\]

\[
= -\frac{\omega_1}{\omega} \ln \left(1 - \frac{\omega}{\omega_1}\right)
\]

**Examples of calculations**

After a starting angle of \(\delta_0 = 2\pi\), a motor is required to attain an angular velocity of 314 rad/s (3000 rpm). What is the necessary power supply voltage for the following motor characteristics:

- Time constant \(\tau = 0.02\) s
- Motor constant \(k = 0.0254\) Vs
- Load torque \(M_L = 4\) mNm
- Frictional torque \(M_f = 0.2\) mNm
- Armature resistance \(R_a = 17.7\) \(\Omega\)

(at 40°C)

First, we calculate \(A\):

\[
A = \frac{\delta_0}{\omega_1} + 1
\]

\[
= \frac{2\pi}{100\pi \times 0.02} + 1 = 2
\]

For \(A = 2\), after interpolation of the table, we obtain: \(\omega/\omega_1 = 0.795\).

This value enables us to calculate the final angular velocity which the motor could theoretically attain:

\[
\omega_1 = \frac{314}{0.795} = 395\text{ rad/s}
\]

\((n_1 = 3770\text{ rpm})\)

In order to attain this final angular velocity \(\omega_1 = 395\) rad/s with the load torque \(M_L\), the motor requires the power supply voltage \(U_0\) during acceleration (see page 5, formula (13)), where \(I = M/L\):

\[
U_0 = \left(\frac{M_L + M_f}{k}\right) T_{\text{max}} + k \times \omega_1
\]

From \(M_L + M_f = 0.004 + 0.0002 = 0.0042\) Nm, we obtain:

\[
U_0 = \frac{0.0042}{0.0254} \times 17.7 + 0.0254 \times 395
\]

\(-2.83 + 10.03 = 13\text{ V}\)

The motor requires 13 V in order to attain an angular velocity of 314 rad/s after a starting angle of \(2\pi\), under a load torque \(M_L = 4\) mNm.
As we have already stated, the motor temperature increases due to the effect of the dissipated power $P_v = PR_m$. In other words, the motor current heats up the armature coil (rotor windings). For the temperature of the motor to increase by a thermal difference $\Delta T$, there must be Joule energy $Q$:

$$ Q = m \times c \times \Delta T $$

in which $m$ represents the mass of the armature (rotor winding) and $c$ the specific heat of copper.

Moreover, part of the heat produced is dissipated through thermal convection, by the air surrounding the armature (rotor). This flow of heat per unit of time $\Delta t$ is basically calculated as follows:

$$ W = \varepsilon \times F \times (T - T_{0}) $$

in which $F$ represents the surface of the armature (rotor), $\varepsilon$ the coefficient of heat conductivity and $T_0$ the ambient temperature.

Since the quantity of heat produced by $P_v$ in a unit of time $\Delta t$ is equal to the sum of the heat energy in the armature and the flow of heat per unit of time $\Delta t$, we obtain:

$$ P_v \times \Delta t = Q + W \times \Delta t $$

or

$$ P_v = m \times c \times \frac{\Delta T}{\Delta t} + \varepsilon \times F \times (T - T_0) $$

This can be transcribed in the form of a differential equation:

$$ m \times c \frac{dT}{dt} + \varepsilon \times F \times (T - T_0) = P_v $$

Generally, the specific heating or thermal resistance is expressed by means of the formula:

$$ R_h = \frac{1}{\varepsilon \times F} $$

and the thermal time constant by means of the formula:

$$ T_{th} = \frac{m \times c}{\varepsilon \times F} = R_h \times m \times c $$

In practice, these two motor parameters can be determined by means of comparatively simple measurements. We thus obtain the thermal differential equation:

$$ T_{th} \frac{dT}{dt} + (T - T_0) = R_h \times P_v $$

If $P_v$ is constant with time, this differential equation is of the same type as that used to describe the behaviour of the motor speed at start (see page 14). Its solution is (provided that $T(t = 0) = T_0$):

$$ T = T_0 + R_h \times P_v \times (1 - e^{-t/T_{th}}) $$

After the time $t >> T_{th}$, the armature windings reach the temperature:

$$ T = T_0 \times R_h \times R_m \tag{25} $$

where the temperature of the motor housing must be taken as the ambient temperature $T_0$ of the armature. $R_h$ and $T_0$ are related in the following way:

$$ T_{th} = m \times c \times R_h \tag{26} $$

In practice, the variation of the rotor temperature with time $T(t)$ is measured with a constant dissipated power $P_v$. The current $I$ must therefore be adjusted to the thermal variation of the resistance of the rotor, so that $P_v$ remains constant while the measurement is being affected:

$$ R_h = R_m (1 + \gamma (T - T_0)) \quad T_0 = 22^\circ C $$

$R_h$ is determined with $T = \infty$ and $T_1$:

$$ R_{h1} = \frac{T - T_1}{P_v} $$

The heating of the housing follows the same thermal function:

$$ T(t) = T_1 + R_{h2} \times P_v \times (1 - e^{-t/T_{th}}) $$

in which $R_{h2}$ is the thermal time constant of the housing (stator) and $R_{h2}$ the relative specific heating (thermal resistance). $T_1$ is the ambient temperature of the air surrounding the motor housing. For an escap® 26.28-213 motor, the values are:

$$ R_m = 5^\circ C / W \quad R_{h2} = 12^\circ C / W $$

$R_{h2}$ was measured with the motor lying flat thermally insulated in still air. Enlarging the surface of the motor housing, e.g. by mounting it on a heat sink, considerably increases the heat flow and reduces $R_{h2}$. If we introduce the values of $R_{h1} = 5^\circ C / W$, $m = 5.3$ g for the mass of copper of the armature and $c = 0.092$ kcal/kg/°C = 365 Joule/kg/°C = 365 Ws/kg/°C, we obtain:

$$ T_{th1} = \frac{m \times c \times R_{h1}}{-0.0053 \times 395 \times 5 - 10.2 \, s} $$

This means that the thermal time constant $T_{th1}$ of the armature is 10 s. By using $c = 0.11$ kcal/kg/°C = 460 Ws/kg/°C for the specific heat of iron and $m = 0.11$ kg for the mass of the housing, we obtain for the thermal time constant of the housing:

$$ R_{h2} = \frac{m \times c \times R_{h2}}{-0.11 \times 460 \times 12 = 607 \, s} $$

The thermal time constant of the motor housing is 10 minutes.

**Calculation examples**

Calculate the rotor temperature of a 23L21-216C motor after 60 s, with a current of 1A (constant dissipated power) at an ambient temperature of 30°C.

The motor is mounted on a metal surface resulting in a thermal resistance $R_{h2}$ (body-ambient) of 10°C/W. $T_1$ and $T_2$ are the thermal time constants of the rotor and of the stator.

$$ T(t) = T_1 + P_v \times [R_{h1} \times (1 - e^{-t/T_{th}})] $$

$$ = 30 + 6.8 \times 7 \times (1 - e^{-60/12}) $$

$$ + 10 \times (1 - e^{0.60/60}) $$

$$ = 30 + 85.6 $$

$$ = 85.6°C $$

60 seconds after switch-on the rotor has reached a temperature of 88°C.

It should be noted that with thermal calculations a safety margin of at least 10% should always be included, as the values of thermal resistances depend on the actual temperature of all parts and on the detailed cooling conditions in the actual application.

Calculate the maximum power dissipation tolerated continuously for an escap® 23L21-216C motor, without heatsink, at an ambient temperature of 40°C.

From $T = T_1 + R_h \times P_v$ one obtains $P_v = (T - T_1)/R_h$

$$ T = 100 \, °C $$

$T_1 = 40\, °C$

$$ R_h = R_{h1} + R_{h2} = 7 + 10 $$

$$ = 23°C/W $$

Then $P_v = 100 - 40/23 = 2.61 \, W$
Or

\[ P_v = I^2 R_m, \text{ with } R_m \text{ being the armature resistance at } 100^\circ C. \]

\[ R_m = R_{22} \times (1 + 0.004 \times 78) = 6.6 \times 1.31 = 8.66 \Omega \]

Then \( I_{\text{max}} = \sqrt{\frac{P_v}{R_m}} = \sqrt{\frac{2.81}{8.66}} = 0.55 \text{ A} \)

This gives a maximum continuous torque of 0.55 \( \times \) 12.4 = 6.8 mNm.

\[ R_{112} \text{ can be reduced by mounting the motor on a metal surface acting as a heatsink. Then the motor can deliver more torque without the armature temperature exceeding } 100^\circ C. \]

Quite often, motors are driven with constant current, primarily through the use of a chopper. As the rotor resistance increases with rising temperature, power dissipation increases as well and the final armature temperature \( T_\infty \) must be calculated differently.

Let \( P_v = I^2 \times R_{112} \) be the dissipated power with the armature at ambient temperature. At \( T_\infty \), the armature resistance will reach a value of

\[ R_m = R_{112} (1 + \gamma \times \Delta T), \text{ where} \Delta T = T_\infty - T_0 \]

Then

\[ P_v = I^2 \times R_m = I^2 \times R_{112}(1 + \gamma \times \Delta T) = P_v (1 + \gamma \times \Delta T) \]

(27)

From the temperature equation (25) follows:

\[ T_\infty - T_0 = R_{112} \times P_v \]

or

\[ P_v = \frac{T_\infty - T_0}{R_{112}} = \frac{\Delta T}{R_{112}} \]

This is introduced into equation (27):

\[ P_v = \frac{\Delta T}{R_{112}} R_{112} = P_v (1 + \gamma \times \Delta T) \]

Then

\[ T = R_{112} \times P_v (1 - \gamma \times R_{112} \times P_v) \]

The final temperature \( T_\infty \) will be

\[ T_\infty = T_0 + \frac{(R_{112} \times P_v)}{1 - \gamma \times R_{112} \times P_v} \]

If the temperature rise due to a constant dissipated power \( P_{vo} \) is defined as

\[ T_0 = R_{112} \times P_{vo} \]

then

\[ T_\infty = T_0 + \frac{\Delta T_0}{1 - \gamma \times \Delta T_0} \]

or

\[ \Delta T = \frac{\Delta T_0}{1 - \gamma \times \Delta T_0} \]

This clearly shows the difference between the temperature rise at constant current and the temperature rise \( T_0 \) at constant dissipated power \( P_{vo} \).

The difference is illustrated by the following example:

An escap\textsuperscript{r} motor type 28DT12-222E is drawing a current of 1A. Ambient temperature is 22°C, the total thermal resistance be 10°C/W.

At 22°C the rotor resistance is 6.2Ω.

Then power dissipation is

\[ P_{vo} = 1^2 \times 6.2 = 6.2 \text{ W} \]

If this dissipated power is held constant, the temperature rise of the armature is

\[ T_0 = 6.2 \times 10 = 62°C \]

If the current was held constant at 1A, the temperature rise would be

\[ T = 62 / (1 - 0.004 \times 62) = 82.5°C \]

At an ambient temperature of 22°C, the armature will reach a final temperature of 84°C with constant dissipated power and 104.5°C with constant current. Note that the maximum permissible rotor temperature of this motor is 158°C.
Solution of the differential equation of movement (page 8)

\[ \frac{d\omega}{dt} + \omega = \omega_1 \]

\( \omega_1 \) being constant with time (i.e. \( \frac{d\omega_1}{dt} = 0 \)), one particular solution of the above equation is \( \omega = \omega_1 \).

The complete solution is obtained by adding the solution of the homogeneous differential equation:

\[ \frac{d\omega}{dt} + \omega = 0 \]

which is easily integrated by separating the variables.

Follows:

\[ \int \frac{d\omega}{\omega} = -\int \frac{dt}{\tau} \quad \text{or:} \]

\[ \ln \omega = -\frac{t}{\tau} + \ln C_1 \]

We thus get the homogeneous solution

\( \omega = C_1 \times e^{-t/\tau} \)

and the complete solution

\[ \omega = C_1 \times e^{-t/\tau} + \omega_1 \]

C_1 is determined by the initial starting condition \( \omega (t = 0) = 0 \).

If \( t = 0 \), we have \( \omega = \omega_1 + C_1 = 0 \),
or \( C_1 = -\omega_1 \).

Therefore the solution of our differential equation is:

\[ \omega = \omega_1 (1 - e^{-t/\tau}) \]

Solution of the differential equation of temperature rise (page 12)

\[ \tau_e \frac{dT}{dt} + (T - T_1) = R_m \times P_v \]

Writing this equation in a different form:

\[ \tau_e \frac{dT}{dt} + T = T_1 - R_m \times P_v \]

and assuming that \( P_v \) be constant with time, we can see that

\( T = T_1 + R_m \times P_v \)

is a particular solution to this equation. Once again we obtain the complete solution by adding the homogeneous equation:

\[ \tau_{th} \frac{dT}{dt} + T = 0 \]

which can easily be integrated by separation of the variables:

\[ \int \frac{dT}{T} = -\int \frac{dt}{\tau_{th}} \quad \text{and} \]

\[ \ln T = -\frac{t}{\tau_{th}} + \ln C_2 \]

or

\[ T = C_2 \times e^{-t/\tau_{th}} \]

Adding this solution to the particular one gives the complete solution:

\[ T = T_1 + R_m \times P_v + C_2 \times e^{-t/\tau_{th}} \]

The initial starting condition \( T (t = 0) = T_1 \) determines the value of \( C_2 \):

\[ T (t = 0) = T_1 + R_m \times P_v + C_2 = T_1 \]

thus

\[ C_2 = -R_m \times P_v \]

The complete solution under the initial starting condition \( T (t = 0) = T_1 \) is therefore:

\[ T = T_1 + R_m \times P_v (1 - e^{-t/\tau_{th}}) \]